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Membrane finite element method for simulating fluid flow in porous medium

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Abstract: A new membrane finite element method for modeling fluid flow in a porous medium is presented in order to quickly and accurately simulate the geo-membrane fabric used in civil engineering. It is based on discontinuous finite element theory, and can be easily coupled with the normal Galerkin finite element method. Based on the saturated seepage equation, the element coefficient matrix of the membrane element method is derived, and a geometric transform relation for the membrane element between a global coordinate system and a local coordinate system is obtained. A method for the determination of the fluid flux conductivity of the membrane element is presented. This method provides a basis for determining discontinuous parameters in discontinuous finite element theory. An anti-seepage problem regarding the foundation of a building is analyzed by coupling the membrane finite element method with the normal Galerkin finite element method. The analysis results demonstrate the utility and superiority of the membrane finite element method in fluid flow analysis of a porous medium.

Key words: *membrane finite element; normal Galerkin finite element method; coupling; fluid flow in porous medium*

1 Introduction

Geo-membranes and membrane materials are used in hydraulic engineering, waterway engineering, hydropower generation, highways, railways, seaports, architecture, mining and military projects because of the anti-seepage effect, light weight, easy roll paving, quick construction, lesser investment, good project quality, deformation adaptation and corrosion-proof properties associated with them (Frind and Pinder 1973; Zhan et al. 1999a, 1999b). However, numerical simulation of geo-membranes in groundwater analysis is very difficult. Many methods of simulating friction elements were developed in different forms based on the discontinuous displacement model (Goodman contact surface). But studies on geo-membranes using experimental results and numerical simulations are rarely conducted, even though geo-membranes are widely used in civil engineering. For the composite geo-membrane, experiments conducted by Liu (2004) showed that the key factors in seepage discharge were pore size, pore shape in a geo-membrane, and hydraulic head. These

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experiments also showed that the type and thickness of the geo-membrane had no effect on seepage discharge. In previous studies, numerical simulations of the geo-membrane were based on Darcy's Law and the normal Galerkin finite element theory. A fine mesh is usually used in traditional fluid flow simulation for complex dam foundation engineering. The thickness of the membrane is very small (usually less than 1 mm), so the amount of calculation is too large to implement. Furthermore, there are lots of singular elements that increase the difficulty of mesh pretreatment, affecting calculation accuracy directly (Neuman 1973; Nayroles et al. 1992; Lu et al. 1994; Zhang et al. 2001). A new discontinuous membrane finite element method, originating from the Goodman contact surface element, was developed to simulate the discontinuity of hydraulic head on both sides of the geo-membrane. The selection and physical meanings of parameters in the discontinuous model are very important both in discontinuous elements of seepage and discontinuous friction elements. These parameters were analyzed based on their physical background. The discontinuous element described in this paper is a kind of element pattern and is part of the finite element method, while the general discontinuous finite element is a numerical method that is different from the conventional finite element (Yan 2006; Zheng and Chen 2008; Luo and Feng 2008). An anti-seepage problem regarding the foundation of a building was analyzed by integrating the membrane finite element method with the normal Galerkin finite element method. The discontinuous membrane finite element method was use to calculate the seepage discharge with various damage rates of the geo-membrane. The simulation results show that this method can simulate geo-membranes quickly and accurately in complex seepage studies.

2 3D membrane finite element model

2.1 Flow governing equation and seepage matrix of 3D membrane finite element method

The well-known basic equation for saturated seepage in a porous medium is as follows:

$$K_x \frac{\partial^2 H}{\partial x^2} + K_y \frac{\partial^2 H}{\partial y^2} + K_z \frac{\partial^2 H}{\partial z^2} = 0 \qquad (1)$$

where K_x, K_y, and K_z are the permeability coefficients in the *x*, *y*, and *z* directions, respectively, and H is the hydraulic head.

In practical hydraulic engineering, some impervious structures (such as impervious curtains, impervious walls and blankets in hydroelectric engineering) are used to control fluid flow in porous formations and fractured rock masses. The permeability coefficients of these impervious structures are much lower than those of ambient foundation materials, usually 1/1 000 to 1/100 of the foundation material permeability coefficient values. When a geo-membrane is used as an impervious structure (such as a level blanket and vertical impervious body with a low height) in hydraulic engineering, the head loss occurs within the extremely thin membrane. This engineering problem can be abstracted as a discontinuous problem in mathematics.

The membrane element method was developed from the Goodman element, which simulates the discontinuous deformation of joints and faults in geotechnical engineering. A membrane element with zero thickness still has eight nodes, and has permeability in the tangential direction but no permeability in the normal direction, as shown in Fig. 1. Nodes 1 through 4 are on the bottom surface and nodes 5 through 8 are on the top surface. The normal hydraulic heads of the bottom and top surfaces are described by the following linear functions, respectively:

$$H_{\mathrm{b}} = \sum_{i=1}^{4} N_i H_i \tag{2}$$

$$H_{\mathrm{t}} = \sum_{i=1}^{4} N_i H_{i+4} \tag{3}$$

where H_i ($i = 1, 2, \cdots, 8$) is the hydraulic head of the ith node of the membrane element, and N_i is the interpolating shape function, which adopts the linear interpolation formula of 2D isoparametric elements:

$$N_i = \frac{1}{4}\left(1+\xi_i \xi\right)\left(1+\eta_i \eta\right) \qquad \left(i = 1, 2, 3, 4\right) \tag{4}$$

where ξ_i is the local coordinate of the ith node of the isoparametric element in the ξ direction, and η_i is the local coordinate of the ith node of the isoparametric element in the η direction. Substituting Eq. (4) into Eqs. (2) and (3), the difference in hydraulic head between the top and bottom surfaces of the element is

$$H_{\mathrm{t}} - H_{\mathrm{b}} = \sum_{i=1}^{4} N_i \left(H_{i+4} - H_i\right) \tag{5}$$

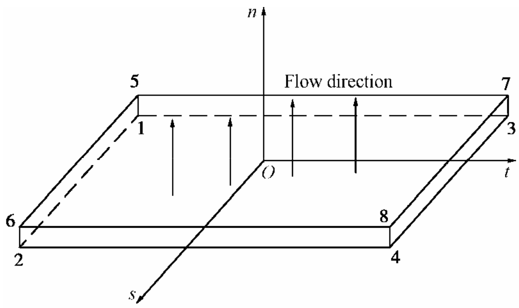

Fig. 1 Sketch of membrane finite element

For unity in mathematics, the following interpolation function is introduced:

$$M_i = -N_i \qquad \left(i = 1, 2, 3, 4\right) \tag{6}$$

$$M_{i+4} = N_i \qquad \left(i = 1, 2, 3, 4\right) \tag{7}$$

Then, Eq. (5) is re-expressed as

$$\Delta H_{s,t} = H_{\mathrm{t}} - H_{\mathrm{b}} = \sum_{j=1}^{8} M_j H_j \tag{8}$$

Based on Darcy's law, fluid velocity can be expressed as $V_{s,t} = K\Delta H_{s,t}/L$, where K is the permeability coefficient along the normal direction of the central plane of the discontinuous thin element, and L is the seepage path length of a general thin element. The traditional method cannot be used to analyze the fluid velocity if $L = 0$. In order to eliminate the thickness effect, we construct the formula between the permeability coefficient, head difference and velocity as follows:

$$V_{s,t} = K_{\mathrm{n}} \Delta H_{s,t} \tag{9}$$

where K_{n} is the flux conductivity of the membrane element (s^{-1}).

According to the following expression of the equivalent discharge at the ith node,

$$Q_i = \iint_{\Omega_e} K_n \Delta H_{s,t} M_i \mathrm{d}s\mathrm{d}t = \iint_{\Omega_e} K_n M_i \left(\sum_{j=1}^{8} M_j H_j \right) \mathrm{d}s\mathrm{d}t = \sum_{j=1}^{8} K_{ij} H_j \tag{10}$$

where Ω_e is the element zone, we can obtain the element seepage matrix $\boldsymbol{K} = \left(K_{ij}\right)_{8\times8}$, where K_{ij} can be expressed as

$$K_{ij} = \iint_{\Omega_e} M_i K_n M_j \mathrm{d}s\mathrm{d}t \tag{11}$$

We can also obtain the following equivalent discharge column matrix:

$$\boldsymbol{Q} = \boldsymbol{KH} \tag{12}$$

where $\boldsymbol{Q} = \left(Q_1, Q_2, \cdots, Q_8\right)^{\mathrm{T}}$ is the equivalent discharge column matrix in the local coordinate system, and $\boldsymbol{H} = \left(H_1, H_2, \cdots, H_8\right)^{\mathrm{T}}$ is the hydraulic head column matrix in the local coordinate system.

The membrane element method can easily be coupled with the normal Galerkin finite element method based on the equivalent nodal discharge.

2.2 Transformation between global coordinate system and local coordinate system

The local coordinate system is used to calculate the element seepage matrix of the membrane. Because the local coordinate system of the membrane is usually different from the global coordinate system, transformation between these two coordinate systems is necessary during assembly of the global seepage matrix by the element seepage matrix. These two coordinate systems are shown in Fig. 2. Direction cosines of local coordinate axes (s, t, n) in the global coordinate system are shown in Table 1.

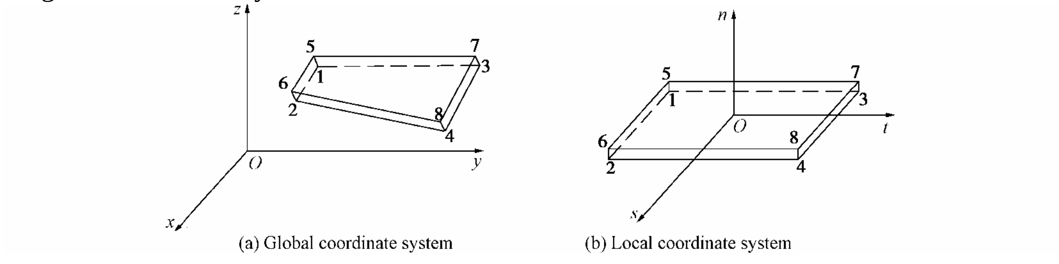

Fig. 2 Sketch of local coordinate system and global coordinate system of geo-membrane element

Table 1 Direction cosines of local coordinate axes in global coordinate system

Local coordinate axis	Direction cosine		
	x	y	z
s	l_1	m_1	n_1
t	l_2	m_2	n_2
n	l_3	m_3	n_3

The transformation between the local coordinate system and the global coordinate system can be expressed as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} s \\ t \\ n \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} s \\ t \\ n \end{bmatrix} \tag{13}$$

Let

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{A} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boldsymbol{A} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{A} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boldsymbol{A} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boldsymbol{A} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{A} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{A} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{A} \end{bmatrix} \tag{14}$$

then the conversion relations for the equivalent discharge column matrix and hydraulic head column matrix between the local coordinate system and the global coordinate system are as follows:

$$\boldsymbol{Q}^{\mathrm{e}} = \boldsymbol{BQ} \tag{15}$$

$$\boldsymbol{H}^{\mathrm{e}} = \boldsymbol{BH} \tag{16}$$

where $\boldsymbol{Q}^{\mathrm{e}} = \left(Q_1^{\mathrm{e}}, Q_2^{\mathrm{e}}, \cdots, Q_8^{\mathrm{e}}\right)^{\mathrm{T}}$ and $\boldsymbol{H}^{\mathrm{e}} = \left(H_1^{\mathrm{e}}, H_2^{\mathrm{e}}, \cdots, H_8^{\mathrm{e}}\right)^{\mathrm{T}}$ are the equivalent discharge column matrix and hydraulic head column matrix in the global coordinate system, respectively. Substituting Eqs. (15) and (16) into Eq. (12), we obtain the following relation between the equivalent discharge column matrix and the hydraulic head column matrix in the global coordinate system:

$$\boldsymbol{Q}^{\mathrm{e}} = \boldsymbol{BKB}^{-1}\boldsymbol{H}^{\mathrm{e}} \tag{17}$$

The relation between the element seepage matrix $\boldsymbol{K}$ in the local coordinate system and the element seepage matrix $\boldsymbol{K}^{\mathrm{e}}$ in the global coordinate system is as follows:

$$\boldsymbol{K}^{\mathrm{e}} = \boldsymbol{BKB}^{-1} \tag{18}$$

3 Seepage mechanism of damaged membrane

In hydraulic engineering, orifice flow will form in pores of the geo-membrane under the head difference ΔH if the geo-membrane is damaged. Based on experiments by Brown, the orifice flow equation can be used to calculate the seepage discharge of the damaged membrane if the permeability coefficient of ambient supporting soil is larger than 0.1 cm/s. The formula of submersed orifice flow is modified to calculate the seepage discharge of the damaged geo-membrane as follows:

$$Q_{\mathrm{m1}} = \xi_{\mathrm{k}} \mu A_1 \sqrt{2g\Delta H} \tag{19}$$

where Q_{m1} is the seepage discharge of the damaged geo-membrane; μ is the discharge coefficient, which is equal to 0.6 here; A_1 is the damaged area of the geo-membrane; g is the acceleration due to gravity; and ξ_{k} is the correction value of the discharge coefficient, defined as

$$\xi_{\mathrm{k}} = \varphi \left(d_{20} \right)^{\alpha} \left(D_{\mathrm{k}} \right)^{\beta} \tag{20}$$

where d_{20} is the grain size of the cushion, D_{k} is the pore diameter of the geo-membrane, and φ, α, and β are coefficients determined by experiments.

The permeability coefficient of the intact geo-membrane is so small that it can be considered an insulate material, therefore, the seepage discharge of the intact geo-membrane can be ignored:

$$Q_{\mathrm{m2}} = m \Delta H A_2 = 0 \tag{21}$$

where Q_{m2} is the seepage discharge of the intact geo-membrane, A_2 is the undamaged area of the geo-membrane, and m is the discharge coefficient of the geo-membrane.

Combining Eq. (19) and Eq. (21), and using the principle of discharge equivalence, we have

$$Q_{\mathrm{m}} = Q_{\mathrm{m1}} + Q_{\mathrm{m2}} = \xi_{\mathrm{k}} \mu A_1 \sqrt{2g\Delta H} \tag{22}$$

where Q_{m} is the equivalent discharge of the damaged geo-membrane. The equivalent discharge through the whole area of the damaged geo-membrane in unit time can be expressed as

$$Q_{\mathrm{m}} = V \left(A_1 + A_2 \right) \tag{23}$$

where V is the equivalent velocity of the damaged geo-membrane. Based on Eq. (9), V can be expressed as

$$V = K_{\mathrm{m}} \Delta H \tag{24}$$

where K_{m} is the flux conductivity of the damaged geo-membrane. Combining Eq. (22) and Eq. (23), we have

$$\xi_{\mathrm{k}} \mu A_1 \sqrt{2g\Delta H} = V \left(A_1 + A_2 \right) \tag{25}$$

Substituting Eq. (24) into Eq. (25), the normal equivalent permeability coefficient of the damaged geo-membrane is deduced:

$$K_{\mathrm{m}} = \xi_{\mathrm{k}} \mu \sqrt{\frac{2g}{\Delta H}} \frac{A_1}{A_1 + A_2} \tag{26}$$

Thus, the normal equivalent permeability coefficient of a damaged geo-membrane element is

$$K_{\mathrm{m}} = \eta_{\mathrm{k}} n_{\mathrm{f}} \sqrt{\frac{2g}{\Delta H}} \tag{27}$$

where η_{k} is the flux coefficient under seepage conditions, defined as

$$\eta_{\mathrm{k}} = \xi_{\mathrm{k}} \mu = \varphi \mu \left(d_{20} \right)^{\alpha} \left(D_{\mathrm{k}} \right)^{\beta} \tag{28}$$

and n_{f} is the damage rate, defined as

$$n_{\mathrm{f}} = \frac{A_1}{A_1 + A_2} \tag{29}$$

4 Application example

The foundation of a large building was used as an example. The groundwater level is 9.5 m. The foundation is composed of sandy soil, whose permeability coefficient is 1.0×10^{-3} cm/s. According to the design of the project, the groundwater level in the foundation should be controlled at 3.0 m or lower. The seepage control technique of combining pump drainage and a vertical impervious body (a geo-membrane in this case) was adopted to control the groundwater level.

4.1 Computing model and boundary conditions

This simulation example focused on a new seepage control method that uses the geo-membrane. A section perpendicular to the excavation boundary on one side of the pit was selected, and a 3D mesh model was set up as shown in Fig. 3. The model had 500 elements and 1 144 nodes. The boundary conditions were as follows: the upstream hydraulic head was 9.5 m, the downstream hydraulic head was 3.0 m, and the possible infiltration boundaries were the partial downstream vertical borders, which were above the downstream hydraulic head.

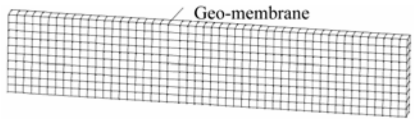

Fig. 3 3D mesh model with coupled geo-membrane element and normal element

4.2 Calculation and analysis cases

For the above problem, several different cases of seepage control systems were simulated using the membrane finite element method presented in this paper. The geo-membrane will unavoidably be damaged during construction. The flux conductivities of damaged geo-membranes under different damage rates were calculated with the equivalent discharge method described above. Different analysis cases of the geo-membrane with different flux conductivities are described in Table 2.

Table 2 Cases of geo-membrane with different flux conductivities

Case	K_{m} (s^{-1})	Case	K_{m} (s^{-1})
1	1.0×10^{-3}	5	1.0×10^{-7}
2	1.0×10^{-4}	6	1.0×10^{-8}
3	1.0×10^{-5}	7	1.0×10^{-9}
4	1.0×10^{-6}		

4.3 Results and analysis

The developed 3D seepage procedures were used to study different cases. The simulated groundwater levels in the foundation in seven cases are shown in Fig. 4. There are concentrated seepage head falls in the geo-membranes in these different cases. The geo-membrane will not be valuable if it is fully damaged, that is, if its flux conductivity reaches 1.0×10^{-3} s^{-1}. However, when the flux conductivity of the geo-membrane is less than 1.0×10^{-7} s^{-1}, its anti-seepage effect is notable: the groundwater level upstream of the geo-membrane impervious body is almost horizontal, and close to the initial groundwater level (9.5 m); the downstream groundwater level is also horizontal, and close to the design groundwater level (3.0 m); and the differences in groundwater levels between cases 5, 6, and 7 are very small. The fall of the hydraulic head in the geo-membrane increases as the permeability coefficient of the geo-membrane decreases. These simulated results fully show the changes of discontinuous values of membrane elements, and also provide a evaluation method for choosing discontinuous parameters of the discontinuous finite element.

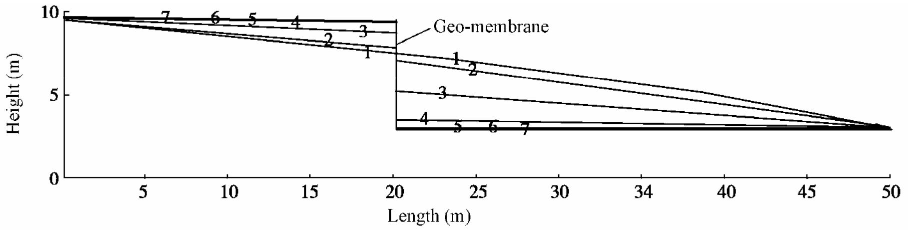

Fig. 4 Simulated groundwater levels in foundation of seven cases

5 Conclusions

Based on discontinuous finite element theory, a new membrane finite element method for simulating fluid flow in a porous medium is presented in this paper. It can be easily coupled with the normal Galerkin finite element method. A seepage control problem regarding the foundation of a large building was analyzed using the membrane finite element method. The simulation results show that the membrane element theory can describe the seepage mechanism of the geo-membrane, thoroughly reflecting the fall of the hydraulic head (a discontinuous characteristic). Moreover, membrane element seepage theory solves the problems that usually appear in the normal Galerkin finite element method. The analysis results show that a geo-membrane will have notable anti-seepage effects if the permeability coefficient of the geo-membrane is 1/10 000-1/1 000 times that of the ambient soil. However, when the permeability coefficient of the geo-membrane is less than 1/10 000 times that of the ambient soil, the difference in the discontinuous water head fall through the membrane element is small.

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